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THE
DOCTRINE

OF

Combinations, Permutations,

AND

COMPOSITIONS,

OF

QUANTITIES,

Clearly and succinctly demonstrated.

By William Emerson.

In tenui labor ————— *VIRG.*

L O N D O N :

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ADVERTISEMENT.

THIS small Treatise was intended to make a part of the Treatise of Algebra; but other matters had swelled the book to too large a size, to admit of this; which therefore was postponed to a further opportunity.

The Subject of it, which is the Doctrine of Combinations and Alternations, is a very entertaining and curious speculation; by which, a great many very delightful problems may be resolved: which will appear strange and surprizing to them that are not versed with these sorts of calculations. The Doctrine thereof is here compleatly delivered, and in a little compass; and the rules demonstrated in the plainest and easiest manner, and therefore I hope it will prove beneficial and acceptable to the reader.



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The Doctrine of COMBINATIONS, &c.

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DEF

Combinations, Permutations, &c.

DEFINITIONS.

DEFIN. I.

THE *Combination* of quantities, is shewing how oft a less number of quantities or things can be taken out of a greater, and combined together, without considering their places, or the order they stand in. This is also called *Election* or *Choice*.

Here every parcel must be different from all the rest, and no two is to have precisely the same quantities or things.

DEF. II.

Permutation, shews how many several ways, the places of any given number of things may be changed. This is also called *Variation*, *Alternation*, and *Changes*.

Here the only thing to be regarded is the order they stand in; for no two parcels are to have all the quantities placed in the same situation.

DEF. III.

Composition, is the taking a given number of quantities, out of as many equal rows of different quantities, one out of every row; and combining them together.

Here no regard is had to their places; and it differs from combination, in which there is but one row of things.

COMBINATIONS, &c.

D E F. IV.

Combinations of the *same form*, are those wherein are the same number of quantities, and the same repetitions. As *abcc*, *bbad*, *cffg*, &c. are of the same form. Also AB^3C^2 , DA^2E^3 , C^3ED^2 , are of the same form; putting the numbers 2, 3 for the repetitions. But *ABBC*, AB^3 , *AACC*, are of different forms.

Here observe, when any quantity is written with an index at the top, it signifies so many repetitions of that quantity, as a^2 signifies *aa*; a^3 signifies *aaa*; and b^4 signifies *bbbb*, &c.

P R O P. I. *Prob.*

To find the number of Permutations or Changes of m things, all different from one another.

R U L E.

Multiply continually, $1 \times 2 \times 3 \times 4 \times 5$, &c. to m terms; and the product is the number of changes.

For 1 thing a , is capable but of one position a . And if there be 2 things a and b ; they are only capable of these 2 variations ab , ba ; whose number is 1×2 .

If there be 3 things a , b , c ; then any 2 of them, leaving out the 3d, will have 1×2 variations; therefore when these 3 are taken in, there will be $1 \times 2 \times 3$ variations.

And if there be 4 things, every 3, leaving out the 4th, will have $1 \times 2 \times 3$ variations. Then taking in successively these four left out, and there will be $1 \times 2 \times 3 \times 4$ variations. And so on as far as you please.

Cor.

PERMUTATIONS.

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Cor. 1. $m \times$ changes of $m-1$ things = changes of m things.

Cor. 2. The changes of $m-1$ things out of m = changes of m things.

For the changes of $abc = 1 \times 2 \times 3$, and taking in d , the changes of any 3 of $abcd = 1 \times 2 \times 3 + 3 \times 1 \times 2 \times 3 = 1 \times 2 \times 3 \times 4$ = changes of $abcd$. And the same of others.

Example 1.

How many changes may be made of the words in this verse? Tot tibi sunt dotes, virgo, quot fydera cœlo.

Ans. $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$.

Example 2.

How many changes may be rung on 6 bells?

Ans. $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$ changes.

Examp. 3.

For how many days can 7 persons be placed in a different position at dinner?

Ans. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$; near 14 years.

PROP. II.

The number of permutations of m things taken n and n , or n at a time; is equal to m changes of $m-1$ things taken $n-1$ at a time.

For suppose these 5 things $abcde$, and first leave out a ; then we shall have the four $bcde$, out of which let there be taken all the 2's, bc , bd , &c. and put v = number of all the variations of every 2, out of the four quantities $bcde$.

Now let a be put in the first place of each of them, which will make abc , abd , &c; then each

B 2

will

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will consist of three letters; and the number of changes will remain the same, whilst a possesses the first place; that is, v = number of variations of every 3 out of the 5 $abcde$, when a is first.

In like manner, if b, c, d, e be successively left out, the number of the variations of all the 2's will also be v ; and, putting b, c, d, e in the first place to make 3 quantities out of 5, there will still be v variations, as before. But these are all the variations that can happen of 3 things out of 5, when a, b, c, d, e are successively put first; and therefore the sum of all these is the sum of all the changes of 3 things out of 5. But the sum of these is so many times v as is the number of things; that is, $5v$ or mv = all the changes of 3 things out of 5. And it is evident, the same way of reasoning may be applied to any numbers, m, n .

P R O P. III. Prob.

Any number m of different things being given; to find how many permutations or changes can be made out of them, taken n by n , or n quantities at a time.

R U L E.

Multiply continually $m \times m-1 \times m-2 \times m-3 \times \&c.$ to n terms, for the changes.

Suppose any number of things $abcdefg$; here $m = 7$, and let $n = 3$. Subtract 1 from 3 and there remains 2, subtract 2 from 7 and there remains 5. Then set the numbers as follows,

n		m
1	—	5
2	—	6
3	—	7

Here it is evident, that the number of changes, made 1 by 1 out of 5 things, will be $5 = v$.

Then

PERMUTATIONS

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Then (Prop. II.) when $m = 6$, $n = 2$, the number of changes $= mv = 6 \times 5 = v$, a second time.

Again, when $m = 7$, $n = 3$, the number of changes $= mv = 7 \times 6 \times 5$, that is, $= m \times m - 1 \times m - 2$, continued to 3 or n terms. And the like may be shewn for any other numbers.

Cor. 1. If $V =$ all the variations or changes of m things, taken n at a time; then, $m - n \times V =$ all the changes of m things, taken $n + 1$ at a time.

For when $m = 7$, $n = 3$, $V = 7 \times 6 \times 5$, and $m - n = 4$. But by this Prop. making $n = 4$, then the changes will be $= 7 \times 6 \times 5 \times 4$; that is, $= V \times m - n$. And so for all other numbers.

Cor. 2. $n V = m \times$ changes of n things out of $m -$ the changes of $n + 1$ things out of m .

For $mV = m \times$ changes of n things out of m ; from which subtracting $m - n$. V (Cor. 1.) gives $n V$ as above.

Ex. 1.

How many changes can be rung with three bells out of eight?

Ans. $8 \times 7 \times 6 = 336$.

Ex. 2.

How many words can be made with 5 letters of the alphabet, admitting all consonants?

Ans. $24 \times 23 \times 22 \times 21 \times 20 = 5100480$.

PROP. IV. Prob.

Any number of things m , being given; whereof there are p things of one sort, and q things of another, &c. To find how many changes can be made out of them all?

RULE.

Multiply and divide according to this formula.

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \dots \text{to } m}{1 \times 2 \times 3 \times \dots \text{to } p \times 1 \times 2 \times 3 \times \dots \text{to } q \text{ \&c.}}$$

For suppose ab two quantities, then there will be two changes, ab , ba . But if for b , you put a , then aa admits but of one alternation $= \frac{2.1}{2.1} = 1$. For ab and ba , being two variations, become aa and aa , which therefore will be but one.

In like manner abc affords six variations, but if it be reduced to aaa or a^3 , then these six variations become but one $= \frac{1 \times 2 \times 3}{1 \times 2 \times 3} = 1$. And if abc be reduced to aac , the six variations will be reduced to only these three, aac , caa , aca ; but half so many as before $= \frac{1 \times 2 \times 3}{1 \times 2} = 3$.

In like manner, if all these be different $abcd$, there will be 24 variations; but if they be all the same, as $aaaa$ or a^4 , then all these 24 will be but one $= \frac{1.2.3.4}{1.2.3.4}$. And if they be reduced to $aaab$, three of them being the same, there will but be four variations $aaab$, $aaba$, $abaa$, and $baaa = \frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3} = 4$. And if the four quantities be

$aabc$, the 24 will be reduced to 12 $= \frac{1 \times 2 \times 3 \times 4}{1 \times 2} = 12$.

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$= 12$; for it was shewn before that aab is reduced to 3, and since c may have four positions among the quantities aab ; therefore the variations of all four $aabc$, will be 4×3 or 12.

After the same manner if we have $abbccc$, if they are all different, they will admit of 720 changes, but because b occurs twice, and c thrice, it must be divided by 2, and then by 6, and the quotient will be $60 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 1 \times 2 \times 3}$, as is easily tried. And the like may be shewn of any other such quantities.

Cor. Hence if any index p occurs twice or oftener, the series must be divided by 1.2.3 ... to p twice, or as oft as that index occurs, in different quantities.

For in that case we shall have a^3b^3 , or $aaa bbb$, where p is 3. Or a^4b^4 , where p is 4.

Examp. 1.

How many variations may be made of the letters in the word Bacchanalia?

Ans. It will be in this form $abcdeffgggg$, and the changes will be

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11}{1 \times 2 \times 1 \times 2 \times 3 \times 4} = 831600.$$

Ex. 2.

How many different numbers will the figures following make, 1220005555?

Ans. $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 1 \times 2 \times 3 \times 1 \times 2 \times 3 \times 4} = 12600.$

P E R M U T A T I O N S.

P R O P. V. *Prob.*

To find the number of permutations of m things, taken n at a time; in which there are p things of one sort, q things of another, &c.

R U L E.

First find all the different forms of combination of these m things, taken n at a time: and then how many combinations there are of each form.

Then by Prop. IV. find the number of changes in any form, which multiply by the number of combinations in that form.

Repeat the same for every distinct form; and the sum of all the products gives the whole sum of the changes required; as is evident from the nature of the Problem.

All the difficulty here is to find the combinations required; and this is easily done, when the quantities are but few, by joining together every two, every three, &c. till you come at the number n ; and then reckoning how many different forms there are, having n at a time, and how many in each form.

But when the quantities are many and often repeated, this method would be impracticable; and then we must have recourse to Prop. VIII. following to find these combinations.

Examp. I.

How many alternations or changes can be made of every four letters out of these eight, aaabbbcc?

Set them down as if you were going to multiply them, and all the products accordingly; omitting such as happen to be the same as some of the foregoing, and such as are out of the question.

a b c

PERMUTATIONS.

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<i>a</i>	<i>b</i>	<i>c</i>							
<i>a</i>	<i>b</i>	<i>c</i>							
<i>aa</i>	<i>ab</i>	<i>ac</i>	<i>bb</i>	<i>bc</i>	<i>cc</i>				
<i>a</i>	<i>b</i>	<i>c</i>							
<i>aaa</i>	<i>aab</i>	<i>aac</i>	<i>abb</i>	<i>abc</i>	<i>acc</i>	<i>bbb</i>	<i>bbc</i>	<i>bcc</i>	
<i>a</i>	<i>b</i>	<i>c</i>							
<i>aaab</i>	<i>aaac</i>	<i>aabb</i>	<i>aabc</i>	<i>aacc</i>	<i>abbb</i>	<i>abbc</i>			
<i>abcc</i>	<i>bbbc</i>	<i>bbcc</i>							

Here we have four of this form $aaab$, whose number of changes is 4, by Prop. IV.

And three of this form $aabb$, number of changes 6.

And three of this form $aabc$, number of changes 12.

Therefore $4 \times 4 = 16$

$$3 \times 6 = 18$$

$$3 \times 12 = 36$$

Number of changes 70

Examp. 2.

How many changes can be made of eight letters out of these ten; $a^4 b^2 c^2 d e$?

Here the sum of the indices must always be 8. And all the forms being found by Prop. VIII. following, and the combinations in each; put them down as in the following table, writing the indices of the quantities for the repetitions. Then find the changes (by Prop. IV.) for every eight, in these different forms, and multiply them severally by the number of combinations in each form, as in the last column.

Forms.

PERMUTATIONS.

Forms.	Numb. Combi.	Chang.	Chang. multip.
422	1	420	420
4211	6	840	5040
41111	1	1680	1680
3221	2	1680	3360
32111	2	3360	6720
22211	1	5040	5040
Sum of all the Changes, 22260			

Ex. 3.

How many different numbers can be made out of 1 unit, two two's, 3 three's, 4 four's, and 5 five's; taken five at a time.

Put a, b, c, d, e for 5, 4, 3, 2, 1; then the things given will be $a^5 b^4 c^3 d^2 e$. Then by Cor. Prop. VIII. following, find all the different forms, these five things are capable of, and by that Prop. the number of combinations in each form; which set down as below. Then the changes in each form, being found by Prop. IV. must be multiplied into these combinations, as follows.

Indices.	Forms.	Combi- nations.
54321	5	1
$m = 15$	41	8
$n = 5$	32	9
$L = 5.$	311	18
	221	18
	2111	16
	11111	1

Then -

PERMUTATIONS.

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Then the number of alternations or changes will be,

$$1 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times 1 = 1 \times 1 = 1$$

$$2 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1} \times 8 = 5 \times 8 = 40$$

$$3 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \times 5}{1 \cdot 2 \cdot 3 \times 1 \cdot 2} \times 9 = 10 \times 9 = 90$$

$$4 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \times 1 \times 1} \times 18 = 20 \times 18 = 360$$

$$5 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \times 1 \cdot 2 \times 1} \times 18 = 30 \times 18 = 540$$

$$6 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \times 1 \times 1 \times 1} \times 16 = 60 \times 16 = 960$$

$$7 \text{ form, } \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1} \times 1 = 120 \times 1 = 120$$

$$\text{Number of all the changes, } \underline{2111}$$

Note, if 10 be to be taken at a time, the number of the forms of combination would be the same; but the combinations themselves would be different.

PROP. VI.

If C be the number of combinations of m different things taken n at a time; then $\frac{m-n}{n+1}C$ will be the number of combinations, taken $n+1$ at a time.

Let $abcde$ be the different things proposed. Then all the different combinations, taken by one's, by two's, by three's, &c. will be as follows.

a ab

COMBINATIONS.

<i>a</i>	<i>ab</i>	<i>abc</i>	<i>abcd</i>	<i>abcde</i>
<i>b</i>	<i>ac</i>	<i>abd</i>	<i>abce</i>	
<i>c</i>	<i>ad</i>	<i>abe</i>	<i>abde</i>	1
<i>d</i>	<i>ae</i>	<i>acd</i>	<i>acde</i>	
<i>e</i>	<i>bc</i>	<i>ace</i>	<i>bcde</i>	
—	<i>bd</i>	<i>ade</i>	—	
5	<i>be</i>	<i>bcd</i>	5	
	<i>cd</i>	<i>bce</i>		
	<i>ce</i>	<i>bde</i>		
	<i>de</i>	<i>cde</i>		
	—	—		
	10	10		

1. Here the combinations of all the five simple letters, is evidently 5 or m .

2. Then all the quantities $abcde$, being combined with all the same 5, the whole number of combinations of two letters, will be 5 times 5 or 25, but the 5 squares are to be rejected, aa , bb , cc , dd , ee . Then there remains 20, which is 4×5 or $m \times m - 1$, but out of these 20, every one is repeated twice over, as ab and ba , ac and ca , &c. Therefore $m \times \frac{m-1}{2}$, or $C \times \frac{m-1}{n+1}$ will be the

just number of the combinations of 2 things, n being = 1.

3. Then if the last 10 or C , be combined with the 5 $abcde$, there will be produced 5×10 or 50 combinations of 3 letters. And out of these there must be rejected 4 with aa , and as many with bb , cc , dd , ee ; that is, 5×4 or $20 = 2C$. Then there remains 30 combinations, or $m - n \times C$. But each combination is thrice repeated, for three letters; for we shall have abc , bac , cab ; and abd , bad , dab , &c. that is, they are as often repeated as there are letters; therefore taking a third part, $C \times \frac{m-2}{3}$ or

or $C \times \frac{m-n}{n+1}$ = the just number of combinations, in this case = 10.

4. Again, if these last be again combined with the five, we shall have 5×10 , or 50 combinations of four letters, in this example. But out of these, there will be six with *aa*, and as many with *bb*, *cc*, *dd*, *ee*; that is, 6×5 or 30, to be rejected. Then there remains $20 = 2 \times 10 = \frac{m-n}{4} C$. But these contain as many repetitions of each, as there are letters, that is, 4 or $n+1$. For amongst them we have *abcd*, *bacd*, *cabd*, and *dabc*; and so of the rest, therefore we must take only $\frac{1}{4}$ of these; and then we shall have $\frac{m-n}{4} C$ for four quantities

combined, which in this case is $5 = \frac{m-n}{n+1} C$.

5. Then if these five be combined with *abcde*, we shall have 5×5 , or 25 of all; but among them there will be four with *aa*, and as many with *bb*, *cc*, *dd*, *ee*; that is, 4×5 , or 20 to be thrown out; then there remains 5, but all these 5 are but the repetition of the same letters differently placed; and therefore we must take the 5th part, or $\frac{1}{5} = 1$, for the number of combinations $= \frac{m-n}{5} C =$

$\frac{m-n}{n+1} C$, in our example.

And it is evident, that the same reasoning may be applied to any number of distinct things whatsoever. And therefore if *C* be any number of combinations of n quantities, then $\frac{m-n}{n+1} C$ will be combinations for $n+1$ quantities.

PROP. VII. *Prob.*

To find the number of Combinations of m things, all different from one another, taken n at a time.

RULE.

Multiply continually $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ &c. to n terms, for the number of combinations.

For let m be the number of the quantities *abcdef*, &c. then it is plain, that m will be the number of all the one's.

Then let $n = 1$, and $C = m$, and by the last Prop. the combinations of all the two's will be $C \times \frac{m-1}{2} = \frac{m}{1} \times \frac{m-1}{2}$.

Again, let $n = 2$, $C = \frac{m}{1} \times \frac{m-1}{2}$; then by the last Prop. the combinations of all the three's will be $C \times \frac{m-2}{3} = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$.

Also if $n = 3$, $C = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$; then the combinations of all the four's will be $C \times \frac{m-3}{4} = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$.

And so forward as far as you will.

Cor. 1. *The number of combinations of n things, taken out of m , is equal to the uncia or coefficient of the $n + 1^{\text{th}}$ term of a binomial, raised to the m^{th} power.*

For

For $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$ &c. to n terms is the $n+1^{\text{th}}$ term of a binomial; by Cor. 1. Prob. V. Sect. I. Algebra.

Cor. 2. *The number of all the possible combinations, of all the one's, two's, three's, &c. is one less than the sum of all the uncia of a binomial raised to the m^{th} power, and that is equal to $2^m - 1$. This is the number of all the elections.*

For the sum of the coefficients of the m^{th} power = sum of the coefficients of $1 + 1^m = 1 + 1^m = 2^m$.

Cor. 3. *There is the same number of combinations of n things out of m , as of $m - n$ things out of m .*

For there is the same variety of taking n things, as of leaving n things; that is, of taking $m - n$ things.

Cor. 4. *There is but one combination of m things out of m ; whether they be all different, or several of one or more sorts.*

Cor. 5. *The number of permutations of m things taken n at a time = the like number of combinations $\times$ into $1.2.3.4.5$ &c. . . to n terms.*

This is plain from this Prop. and Prop. III.

Ex. 1.

How many combinations can be made of six letters out of ten?

$$\text{Ans. } \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 210.$$

Ex. 2.

How many combinations can be made with three letters of the alphabet?

$$\text{Ans. } \frac{24 \times 23 \times 22}{1 \times 2 \times 3} = 2024.$$

SCHOL.

S C H O L.

The number of divisors of m quantities, $abcd$, &c. is $2^m - 1$. And the number of aliquot parts, $2^m - 2$. For the number of all the combinations of the one's, two's, three's, &c. is the same as the number of the divisors. And the number of aliquot parts is one less; because $abcd$ is a divisor, but not a part of itself.

P R O P. VIII. *Prob.*

To find the number of combinations of m things, taking n at a time; in which there are p things of one sort, q things of another sort, &c.

Here when any quantity is often repeated, as aaa , $bbbb$, &c. they are written thus a^3 , b^4 , &c. where the index shews the number of repetitions, of any quantity, or how often it is to be taken. In the former cases of combination where the quantities are all distinct, the solution is had by a single rule, or a single operation. But this cannot be done where any of the quantities are repeated, for this requires several operations, for the different forms of combination, that n things are capable of, one for each form; whereas if they were all distinct, there could be only one form for all.

To proceed regularly, we must first find how many different forms n things will admit of, and what they are; and then how many combinations are in each form. And the sum of all these combinations will be the number required.

R U L E, *for the number of forms.*

1. Place the m things so, that the greatest indexes may be first, and the rest decrease in order. Then begin at the first letter, and join it with the second,

second, third, fourth, &c. to the last; then take the second letter and join it with the third, fourth, &c. and then take the third letter and proceed the same way; and then the fourth, and the fifth, &c. till all be done; always taking care to reject such combinations as you have had before. Thus you have the combinations of all the two's.

Then for all the three's, join the first letter to every one of the two's, successively; and then the second, and third, &c. to the last; taking care to leave out such combinations as are had before. Then you have all the combinations of the three's.

Proceed the same way to get the combinations of all the four's, all the five's, &c. till you come at n , whose combinations are required. So you will at last get all the several forms of combination of n things; and how many there is in each form.

But in particular cases, shorter ways than this may often be found out.

2. To find the number of combinations in any form; let the quantities, both in the m things, and n things, be ranged according to the indexes, setting the greatest first, as has been said. Then if the quantities are not many, the number of combinations may be found out, and reckoned up, after a like manner, as their different forms were found out before; beginning at the first, and going gradually thro' them all. As if you have $a^3 b^3 c^2 d$; and you would know how many are contained in it, of this form $aabc$. Here aa may be connected with bc , bd , cd ; and there are three such squares which may be connected in like manner, viz. aa , bb , and cc ; therefore 3×3 or 9 will be the number of combinations in that form.

But when there are many terms, we must proceed to calculation in this manner. Let the things given whose number is m , be those $a^p b^q c^r d^s$, &c. where p , q , r , s are the indices or repetitions, no

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matter whether equal or unequal. And let $a^t b^v c^w d^x$, &c. be the form to be combined, the indices being t, v, w, x , &c. either equal or unequal. And when a letter is but once expressed, its index will be 1. Then for the

RULE for any single form.

Let A, B, C, D , &c. be the number of different letters in the m things exposed, whose indices are equal to, or greater than (t, v, w, x , &c. respectively) the indexes in the n things.

Then the number of combinations in that form is $\frac{A \times B - 1 \times C - 2 \times D - 3, \&c.}{RS, \&c.}$ continued to

as many terms as there are different letters in the form proposed; where $R = 1 \times 2$, or $1 \times 2 \times 3$, or $1 \times 2 \times 3 \times 4$, &c. when any index in the form is twice, thrice, or four times, &c. repeated; and the like for S , &c.

3. The same operation is to be repeated for every form; and then the sum of all, gives the number of combinations; being all that n things out of m , can admit of.

For suppose any number of things as $m = a^5 b^3 c^2 d$, and the form proposed. $a^3 b^2 c$

It is evident 3 (the index of a^3) is contained in 5 and 3, (the indexes of $a^5 b^3$), which gives 2 case, where a cube can be taken out of the m things, here $A = 2$.

Then, 2 (the index of b^2) is contained in 5, 3, 2 (the three indexes of $a^5 b^3 c^2$); here $B = 3$. So there are three cases where a square can be taken, considered singly. But as some cube goes along with it, that takes away one case, and there remains $B - 1$ or two cases only. Now for every one of these, there will be A cases, where the cube and square are both contained in the m things: that is,

is, the number of combinations with a cube and a square will be $A \times B - 1$.

Again, 1 (the index of c) is contained in 5, 3, 2, and 1, (the indices of all the quantities $a^5 b^3 c^2 d$); whence $C = 4$; so that there will be four cases, where (c) any simple quantity can be taken. But since a cube and a square is to go along with it; these take up two places, and destroy two of the cases; so that the number of cases where a simple quantity can be connected with a square and a cube (whose places are fixt, or species determined), will be $C - 2$. And for every one of these, there will be $A \times B - 1$ cases by varying the species of the cube and square; so the number of these three combinations, will be $A \times B - 1 \times C - 2$. And in like manner, if there had been a fourth quantity (d) to be combined, we should have $D =$ number of the simple cases, and $D - 3$ the number of cases, when going along with the other three. And then the whole number of combinations of quantities, with four different indices, would be $A \times B - 1 \times C - 2 \times D - 3$, for d must be supposed to have a different index from any of the rest.

It is evident, this way of reasoning is general, and may be applied to any collections of quantities whatever; where the indices of all the quantities n , in the proposed form, are all different. But it matters not, whether the indices in the whole collection of quantities m , which are exposed, be the same or different.

Now when any index in the proposed form happens to be repeated several times, the former number must be divided by a correspondent number, after a like manner as was shewn in Prop. IV. for the changes of things.

Let $m = a^3 b^2 c^2$, the form $a^3 b$, then $A = 3$; $B = 3$, and $A \times B - 1 = 6$. But if the form be

C 2

$a^3 b^2$

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a^2b^2 or ab , in either case A and B remain the same, and $\frac{A \times B - 1}{1.2} = 3$, which shews all the cases that two squares, or two single letters, can be taken out of m . Here $R = 1 \times 2$.

Again, let $m = a^3b^2c^2d$, and the form a^3b^2c ; then $A = 4$, $B = 4$, $C = 4$, and $4.3.2 = 24$, the number of combinations of a cube and a square and a single letter. But if the form be a^3b^2c ; A, B, C will be the same as before, and $\frac{4.3.2}{1.2} = R$

$= 12$ combinations of two squares and a single letter, as is easily tried. And if the form be $a^3b^2c^2$; A, B, C, will still be the same, and then $R = 1 \times 2 \times 3$, whence $\frac{4.3.2}{1.2.3} = 4$, the combina-

tions of three squares; and it is evident by inspection, there can be no more. And there would be the same number for the form abc . And the same will hold in all such quantities, where the indexes are alike. For a repetition of the same index causes a certain number of repetitions among the quantities combined; which ought to be reduced in proportion. And what is said of the repetition of one index, is equally true for the repetition of any other index, and then S comes into the calculation.

Cor. Hence it appears, that before the required number of combinations can be found; all the different forms of combination must be had. And these may be obtained by Art. 1. before; or rather, when the quantities are many, by the following method.

Here we must work by the indices. Put $L =$ number of different letters.

Then set down all the indexes contained in the m quantities, taking the greatest first. Then out of these

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these must be made so many rows of figures as there are different forms, and the sum of the figures in each row must make n , and no row must exceed L places. Now to get all these forms, set down the greatest index first, and the next lesser towards the right hand, and so on. Then proceed from one row to another, by breaking the least numbers (those towards the right) still into lesser numbers, till they be units. Thus proceed gradually figure by figure, from the right towards the left, till all the numbers be diminished as low as they will admit of.

Ex. 1.

Let the things proposed be $a^3b^2c = m$, to find the number of combinations made of every 3 (n) of these quantities.

The indexes 3 2 1 . $L = 3$.

Forms		combin.
3		1
2 1		4
1 1 1		1

Sum of the combinations, 6

In this example there are 3 forms; 1. a cube; 2. a square and a single one; 3. three single ones. The first admits of one combination only; the second of 4, and the third of 1, whose sum is 6.

Ex. 2.

Let $a^3b^3c^2$ be proposed; to find the number of combinations, taken 4 at a time?

Indexes.	Forms.	Combin.
3 3 2	3 1	4
$L = 3$.	2 2	3
	2 1 1	3

Sum of the combin. 10.

C 3

Ex.

COMBINATIONS.

Ex. 3.

How many Combinations, in $a^4b^2c^4de$; taking 8 at a time?

Indexes.	Forms.	Combinations.
42211	422	1
$n = 8$	4211	6
$L = 5$	41111	1
	3221	2
	32111	2
	22211	1

Combinations 13

In this example there is no three's to be combined with the four.

Ex. 4.

Suppose $a^5b^4c^4d^4e^4f^4g$ be proposed; to find all the combinations, taken 10 at a time.

Indices.	Forms.	Combinations.
5544431	55	1
$n = 10$	541	40
$L = 7$	532	40
	5311	100
	5221	80
	52111	100
	511111	12

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Forms.

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Forms.	Combin.	Forms.	Combin.
442	40	3331	80
4411	110	3322	90
433	50	33211	360
4321	400	331111	75
43111	250	32221	180
4222	50	322111	240
42211	300	3211111	30
421111	125	22222	6
4111111	5	222211	45
	1320	2221111	20
			1126
			1320
			373
			2819

To give account of our proceedings in this Example, it may be observed, that all the figures decrease towards the right. In the first place, we take the greatest 5 and 5, and then break the latter 5 into 4 and 1, or into 3 and 2, or into 3 and 1, 1. Having done with 5, 3, we take 5, 2, and fill up (10) with 2, 1, or with 1, 1, 1.

Having done with 5, we take 4, and fill up with the greatest numbers, not exceeding 4; as with 4, 2, and 4, 1, 1. For if we had taken a bigger number than 4, we shall find it dispatched before; and therefore we must always take an equal or a less number. In the next place we take 4, 3, and fill up with 3, or 2, 1, or 1, 1, 1; and thus we go on till all the right hand numbers are dissolved into 1's.

Having finished those with 4, we take 3, and put to it 3,3,1, the greatest numbers that exceed not 3; and proceed as before; but we have now

COMBINATIONS.

additional numbers, to do with, above 2. And 3 being done, we go to 2, and the whole collection ends with 2221111; for we cannot break 2 into 1 and 1, for the number of places would then exceed 7, which it cannot do.

All these forms being thus had, their several combinations are obtained, by the second article, and set against them in the table. And this example may serve as a specimen for other more difficult cases.

SCHOLIUM.

When n , the number of things to be taken, is very great; you may first take the number of $m - n$ things; and find how many combinations there will be for them; for there will be just as many for one as for the other. And then having the combinations of $m - n$ things; by taking these severally out of the whole, you will have the combinations of n things required. As if $m = a^3b^2c$, and $n = 4$, then $m - n = 2$, and all the two's will be aa, ab, ac, bb, bc . These taken severally out of a^3b^2c , will leave $abbc, aabc, aabb, a^2c, a^3b$; which are all the combinations of the four's. But the number of forms will not be the same in both, there being only two forms in the two's; and three forms in the four's. Neither will the number of combinations, in the correspondent forms, be the same.

PROP. IX.

The number of compositions of n things taken out of n rows, each row consisting of m things, is m^n , or the n^{th} power of m .

Let there be any number of rows $a \ b \ c \ d$
as these annexed. It is plain the num- $e \ f \ g \ h$
ber of single things as a, b, c, d , is $i \ k \ l \ m$
 m or m^1 .

Then

Then the number of combinations of every 2, is had by joining each quantity in the second row to all the quantities in the first, which will make as many times m as there are things in the second row, or m times m , that is m^2 ; for all the two's.

Again, taking in, the third row; there will be as many times m^2 , as there are things in the third row, that is m times mm , or m^3 ; for the composition of three things.

After the same manner if a fourth row was taken in; all the combinations of every four, would be m^4 ; and so on. And therefore universally when n rows are taken in, the number of combinations will be m^n , which is the number of compositions.

Cor. 1. *The number of compositions of all the one's, two's, three's, &c. to n , is $\frac{m^n - 1}{m - 1} m$.*

$$\text{For } \frac{m^4 - 1}{m - 1} = m^3 + m^2 + m + 1.$$

$$\text{And } \frac{m^4 - 1}{m - 1} m = m^4 + m^3 + m^2 + m + 1, \text{ as will ap-}$$

pear by division, or in general, $\frac{m^n - 1}{m - 1} m = m^n + m^{n-1} + m^{n-2} + \dots + m + 1$, &c. to $m =$ compositions of all the one's, two's, three's, &c. to n .

Cor. 2. *Hence, may be found the composition of n things out of m , as follows. Involve m to the n^{th} power for the answer?*

Examp. 1.

How many compositions, of 3 letters out of 20?

Ans. $20^3 = 8000$. But these should be here, 3 different alphabets.

Ex. 2.

How many changes are there on throwing four dice?

Ans. $6^4 = 1296$.

P R O P.

COMPOSITION

PROP. X. Prob.

If there be m rows of quantities given, having the same quantities, and the same number of them, as $a, b, c, d, \&c.$ To find the number of compositions of m things taken out of the m rows; for any given form of these quantities, as $a^x b^y c^z d^w, \&c.$

RULE

Put $V =$ variations of the things $a^x b^y c^z d^w,$

And $A =$ variations or alternations of the indexes, *twice*, both found by Prop. IV. Then will

$AV =$ number of compositions required.

For let the different rows in the last $a\ b\ c\ d$
 Prop. be made all alike, or the first row $a\ b\ c\ d$
 m times repeated, as here. And let the $a\ b\ c\ d$
 number of things proposed be $a^x b^y c^z$; and $a\ b\ c\ d$
 first let the index 3 be fixt to a , and 2 to $a\ b\ c\ d$
 b , &c. Now since a^x may be taken out of $a\ b\ c\ d$
 any 3 rows, and b^y out of any two remain- $a\ b\ c\ d$
 ing rows; and c out of the last remaining row;
 therefore there will be so many ways of taking
 these letters, as each of them can be placed in dif-
 ferent rows, that is, in different places or situations;
 for the varying the rows is the same thing as va-
 rying the places of the letters; and consequently
 the number of different ways of taking them out
 of the several rows, is equal to the number of va-
 riations or permutations of these letters, to be
 found by Prop. IV. which number is $= V$. And
 this will hold as long as the indexes are fixt to these
 particular letters, and to none else.

But since in any one form as $a^x b^y c^z$, that form will
 take in as many cases, as there can be variations
 made in shifting the indices, from one letter to a-
 nother; therefore there will be so many times V ,
 as is the number of these variations. Therefore if

$A =$

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A = number of variations or alternations, of the indices 3, 2, 1; or in general, of t, v, x, w . Then **AV** will be the whole number of compositions, that particular form will admit of.

Cor. 1. Hence in any $a^t b^v c^w d^x \&c.$ where the indices are fixt invariably to their particular letters; the number of compositions **V** will be,

$$= \frac{1 \times 2 \times 3 \times 4 \dots \text{to } m}{1.2.3 \dots \text{to } t \times 1.2.3 \dots \text{to } v \times \&c.}$$

Cor. 2. If n = number of different letters in any form $a^t b^v c^w d^x \&c.$ Then the whole number of compositions for that form will be =

$$\frac{1.2.3 \dots \text{to } n}{R S \&c.} \times \frac{1.2.3 \dots \text{to } m}{1.2 \dots t \times 1.2 \dots v \times \&c.}$$

where $R = 1.2$, or $1.2.3 \&c.$ according as any index is twice or thrice, $\&c.$ repeated; and the like for S , and so on.

Here follow some examples.

Examp. 1.

How many compositions are in the form $a^3 b^2 c$?

Here $n = 3, m = 6, t = 3, v = 2, w = 1$. And here is no repetition of indexes. Therefore

$$\text{Ans. } 1.2.3 \times \frac{1.2.3.4.5.6}{1.2.3 \times 1.2} = 6 \times 60 = 360.$$

Ex. 2.

How many compositions in the form $a^3 b^2 c$?

Here $n = 3, m = 5, t = 2, v = 2, w = 1$. The index 2 is twice repeated. Therefore

$$\text{Ans. } \frac{1.2.3}{1.2} \times \frac{1.2.3.4.5}{1.2 \times 1.2} = 3 \times 30 = 90.$$

Ex. 3.

To find the compositions in the form $a^3 b^3$.

Here

COMPOSITION.

Here $n = 2$, $m = 6$, $t, v = 3$, $w = 0$. The index 3 is twice repeated; and the letters a, b , thrice.

$$\text{Ans. } \frac{1.2}{1.2} \times \frac{1.2.3.4.5.6}{1.2.3 \times 1.2.3} = 1 \times 20 = 20.$$

Ex. 4.

To find the number of compositions of the form $a^1 b^1 c^2 d^2$.

Here $n = 4$, $m = 10$, $t, v = 3$, $w, x = 2$. The index 3 is twice repeated, as also the index 2. The letters, a, b , are thrice repeated; and c, d , twice.

$$\text{Ans. } \frac{1.2.3.4}{1.2 \times 1.2} \times \frac{1.2.3.4.5.6.7.8.9.10}{1.2.3 \times 1.2.3 \times 1.2 \times 1.2} = 6 \times 25200 = 151200.$$

Ex. 5.

How many compositions in the form $a^1 b^1 c^1 def$?

Here $n = 6$, $m = 14$; $t = 5$; $v, w = 3$, $x = 1$, &c. Here the index 3 is twice repeated, and the index 1 thrice. The letter a , 5 times repeated; and b and c thrice. Hence,

$$\text{Ans. } \frac{1.2.3.4.5.6}{1.2.3 \times 1.2} \times \frac{1.2.3.4.5.6.7.8.9.10.11.12.13.14}{1.2.3.4.5 \times 1.2.3 \times 1.2.3} = 60 \times 20180160 = 1210809600. \text{ So prodigi-}$$

ously do the numbers increase in these operations.

SCHOL.

It may be observed, that if the multinomial $A + B + C + D$ &c. be raised to the m^{th} power, and the given form, whose number of compositions is sought, be found among the terms of that power; the numeral coefficient annexed to it, will always be the number of variations V , that we want. See my Algebra, Schol. to Prob. LIX. B. I. Thus for the form $a^1 b^2 c$, look in the sixth power, and you'll find $60A^1 B^2 C$; therefore $60 = V$. Also in the fifth

fifth power, you'll find $30A^4B^2C$, therefore $V=30$, for the variations of a^2b^2c . Also with a^2b^2 in the sixth power, you'll find 20 for V ; and so of others. And this being known, this doctrine may be serviceable for finding the unciæ of any term of a multinomial. As suppose you have the 4th term of the 7th power of $A+B+C$ &c. which is A^4B^3 , A^5BC , and A^6D ,

The variations of A^4B^3 is $\frac{1.2.3.4.5.6.7}{1.2.3.4 \times 1.2.3} = 35$.

and the variations of A^5BC = $\frac{1.2.3.4.5.6.7}{1.2.3.4.5} = 42$;

and the variations of A^6D = $\frac{1.2.3.4.5.6.7}{1.2.3.4.5.6} = 7$;

therefore the complete term is $35A^4B^3 + 42A^5BC + 7A^6D$. And the same may be done for any other term, if the letters that compose it be known. The doctrine of Combinations and Permutations, is likewise applicable to many other purposes, particularly to several problems of chance, which for brevity's sake I omit.

F I N I S.

CHRONOLOGY:

the value of the

...and to the ...
...this doctrine ...
...the power ...
...the ...

RECORDS TIME

with power of $A + B + C$ sec. whereas A, B, C are

THE UNIVERSITY OF CHICAGO

The first of these is the fact that the

... to

THE VARIOUS ALLEGATIONS OF THE

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1944

The doctrine of "unborn ones and unborn ones"

is likewise applicable to numerous other purposes, and is especially valuable in the treatment of various forms of

1945

1941

1940
